To solve a system of equations, you can change the equation's matrix into a matrix with echelon form which would lead to a triangular form of equations

eg. $$\left.\begin{array}{ll} x + y - z = 2 & x = \frac{21}{5} \\ 5y + 2z = 10 & y = \frac{4}{5} \\ 3z = 9 & z = 3 \end{array}\right\} \text{triangular form of equations}$$

To change the ~~matr~~ equation's matrix into an echelon form matrix we use the elementary row operations:

1. Interchange 2 rows
2. Multiply a row by non zero constant
3. Add a multiple of a row to another row

Ex. Reduce the following matrix to echelon form

$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{pmatrix} \xrightarrow[\substack{R_2 - 2R_1 \\ R_3 - 2R_1}]{R_4 + R_1} \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \Rightarrow$$

$$\begin{array}{l} R_2 \longleftrightarrow R_3 \\ \frac{1}{8} R_2 \end{array} \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \xRightarrow{R_4 + 3R_2} \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{pmatrix} \Rightarrow$$

$$R_4 - 29R_3 \quad \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \rightarrow \text{echelon form}$$

Note: The row echelon form of a matrix is not unique or there can be many different echelon forms of a matrix

* Matrices A and B are row equivalent matrices if there's a sequence of ~~row~~ elementary row operations that converts A to B / B to A

Theorem: Matrices A and B are row equivalent if and only if they can be reduced to the same row echelon form

Reduced echelon form: It's a matrix in echelon form and with 2 more conditions:

1- The leading entry in each non zero row is ± 1.
2- Each leading entry is the only non zero entry in its column.

$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} \xrightarrow{\frac{1}{24}R_4} \begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 - 5R_4]{R_3 + R_4,\ R_2 + 5R_4} \begin{pmatrix} 1 & 2 & -4 & -4 & 0 \\ 0 & -1 & 10 & 9 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_1 + 4R_3]{R_2 - 10R_3} \begin{pmatrix} 1 & 2 & 0 & -8 & 0 \\ 0 & -1 & 0 & 19 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 0 & 30 & 0 \\ 0 & -1 & 0 & 19 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem: Each matrix is row equivalent to one and only one reduced echelon form

Solve: $x_2 - 4x_3 = 8$

$2x_1 - 3x_2 + 2x_3 = 1$

$5x_1 - 8x_2 + 7x_3 = 1$

Step 1: Augmented matrix $\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right)$

Step 2: get echelon form

$$R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right) \xrightarrow{R_3 - \frac{5}{2}R_1} \left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right) \xrightarrow{R_3 + \frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array}\right)$$

$2x_1 - 3x_2 + 2x_3 = 1$

$x_2 - 4x_3 = 8$

$0 = \frac{5}{2}$ No solution

Theorem: A linear system is consistent if and only if an echelon form of the augmented matrix has no row of the form $(0 \; 0 \; \dots \mid b)$ where b is non zero

*Continue with different matrix $\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$

Step 3: get reduced echelon form

$$\begin{array}{l} R_2 + R_3 \\ R_1 + 2R_3 \end{array} \Longrightarrow \left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \xrightarrow[\frac{1}{2}R_2]{} \left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \xrightarrow{R_1 - 2R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

Step 4: Rewrite the system of equations

$$x_1 + 6x_2 \quad + 3x_4 = 0$$
$$x_3 - 4x_4 = 5$$
$$x_5 = 7$$

infinite number of solutions

Pivot positions

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \qquad \left(\begin{array}{cccccc} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

echelon form reduced echelon form

O → pivot positions

Basic unknowns: x_1, x_3, x_5 these ~~are~~ unknowns are deduced from free unknowns

Free unknowns: x_2, x_4 these unknowns can be replaced by any value and are used to deduce the basic unknowns

Circles are in 1st, 3rd and 5th columns ∴ x_1, x_3 and x_5 are basic unknowns

let $x_2 = t$ $x_4 = s$

$$x_1 = -6t - 3s$$
$$x_3 = 5 + 4s$$